Supplementary materials to Sato and Kunitomo (2026) in JJSD [1]

prepared by

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Lemma A.2 : Let $p_{kj}^{(n)} = \sqrt{\frac{2}{n+\frac{1}{2}}} \cos\left[\frac{2\pi}{2n+1}(k-\frac{1}{2})(j-\frac{1}{2})\right]$ and $p_{kj}^{*(n)} = \sqrt{\frac{2}{n+\frac{1}{2}}} \sin\left[\frac{2\pi}{2n+1}k(j-\frac{1}{2})\right]$ for $k, j = 1, \cdots, n$. Then, for any $a, b = 1, \cdots, n$,

$$\sum_{a=1}^{n}\sum_{j=1}^{n} p_{aj}^{(n)} p_{bj}^{(n)} = \sum_{a=1}^{n}\sum_{j=1}^{n} p_{aj}^{*(n)} p_{bj}^{*(n)} = \sum_{a=1}^{n}\sum_{j=1}^{n} p_{ja}^{*(n)} p_{jb}^{*(n)} = 1 \ ,$$

Proof of Lemma A.2 : For any $a, b = 1, \cdots, n$ the following identities hold :

$$\sum_{j=1}^{n} p_{aj}^{(n)} p_{bj}^{(n)} = \sum_{j=1}^{n} p_{aj}^{*(n)} p_{bj}^{*(n)} = \sum_{j=1}^{n} p_{ja}^{*(n)} p_{jb}^{*(n)} = \delta(a,b) \ ,$$

where $\delta(a,a) = 1$ and $\delta(a,b) = 0$ for any $a \neq b$.

[1] 2026-10-30. Supplementary materials to Sato and Kunitomo (2026, SK (2026)) in JJSD.

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(I) First, consider the case $a \neq b$. We use the following identities :

$$
\begin{aligned}
& \frac{4}{2n+1}\sum_{j=1}^{n}\cos\frac{2\pi}{2n+1}(a-\frac{1}{2})(j-\frac{1}{2})\cos\frac{2\pi}{2n+1}(b-\frac{1}{2})(j-\frac{1}{2}) \\
= & \frac{1}{2n+1}\sum_{j=1}^{n}\left[\left(e^{i\frac{2\pi}{2n+1}(a-\frac{1}{2})(j-\frac{1}{2})}+e^{-i\frac{2\pi}{2n+1}(a-\frac{1}{2})(j-\frac{1}{2})}\right)\left(e^{i\frac{2\pi}{2n+1}(b-\frac{1}{2})(j-\frac{1}{2})}+e^{-i\frac{2\pi}{2n+1}(b-\frac{1}{2})(j-\frac{1}{2})}\right)\right] \\
= & \frac{1}{2n+1}\sum_{j=1}^{n}\left[e^{i\frac{2\pi}{2n+1}(a+b-1)(j-\frac{1}{2})}+e^{i\frac{2\pi}{2n+1}(a-b)(j-\frac{1}{2})}+e^{-i\frac{2\pi}{2n+1}(a+b-1)(j-\frac{1}{2})}+e^{-i\frac{2\pi}{2n+1}(a-bl)(j-\frac{1}{2})}\right] \\
= & \frac{1}{2n+1}\left[e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}\frac{1-e^{i\frac{2\pi}{2n+1}n(a+b-1)}}{1-e^{i\frac{2\pi}{2n+1}(a+b-1)}}+e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}\frac{1-e^{-i\frac{2\pi}{2n+1}n(a+b-1)}}{1-e^{-i\frac{2\pi}{2n+1}(a+b-1)}}\right. \\
& \qquad \left.+\left[e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-l)}\frac{1-e^{i\frac{2\pi}{2n+1}n(a-b)}}{1-e^{i\frac{2\pi}{2n+1}(a-b)}}+e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}\frac{1-e^{-i\frac{2\pi}{2n+1}n(a-b)}}{1-e^{-i\frac{2\pi}{2n+1}(a-b)}}\right]\right] \\
= & \frac{1}{2n+1}\left[\frac{1-e^{i\frac{2\pi}{2n+1}n(a+b-1)}}{e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}-e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}}+\frac{1-e^{-i\frac{2\pi}{2n+1}n(a+b-1)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}}\right. \\
& \qquad \left.+\frac{1-e^{i\frac{2\pi}{2n+1}n(a-b)}}{e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}-e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}}+\frac{1-e^{-i\frac{2\pi}{2n+1}n(a-b)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}}\right] \\
= & \frac{1}{2n+1}\left[\frac{e^{i\frac{2\pi}{2n+1}n(a+b-1)}-e^{-i\frac{2\pi}{2n+1}n(a+b-1)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}}+\frac{e^{i\frac{2\pi}{2n+1}n(a-b)}-e^{-i\frac{2\pi}{2n+1}n(a-b)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}}\right] .
\end{aligned}
$$

Since $2\pi n/(2n+1) = \pi - \pi/(2n+1)$,

$$
\begin{aligned}
& \frac{1}{2n+1}\left[\frac{e^{i\pi(a+b-1)}e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}-e^{-i\pi(a+b-1)}e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}}\right. \\
& \qquad \left.+\frac{e^{i\pi(a-b)}e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}-e^{-i\pi(a-b)}e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}}\right] .
\end{aligned}
$$

When $a+b$ is odd, then $a-b$ is odd. Then $e^{i\pi(a+b-1)} = e^{-i\pi(a+b-1)} = 1$, $e^{i\pi(a-b)} = e^{-i\pi(a-b)} = -1$ and hence the expression is zero. When $a+b$ is even, $a-b$ is even. Therefore, $e^{i\pi(a+b-1)} = e^{-i\pi(a+b-1)} = -1$, $e^{i\pi(a-b)} = e^{-i\pi(a-b)} = 1$ and hence the expression is zero.

When $a-b=0$,

$$
\begin{aligned}
& \frac{1}{2n+1}\sum_{j=n}^{n}\left[e^{i\frac{2\pi}{2n+1}(a+b-1)(j-\frac{1}{2})}+e^{i\frac{2\pi}{2n+1}(a-b)(j-\frac{1}{2})}+e^{-i\frac{2\pi}{2n+1}(a+b-1)(j-\frac{1}{2})}+e^{-i\frac{2\pi}{2n+1}(a-b)(j-\frac{1}{2})}\right] \\
= & \frac{1}{2n+1}\left[e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}\frac{1-e^{i\frac{2\pi}{2n+1}n(a+b-1)}}{1-e^{i\frac{2\pi}{2n+1}(a+b-1)}}+e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}\frac{1-e^{-i\frac{2\pi}{2n+1}n(a+b-1)}}{1-e^{-i\frac{2\pi}{2n+1}(a+b-1)}}+2n\right] \\
= & \frac{1}{2n+1}\left[\frac{1-e^{i\frac{2\pi}{2n+1}n(2a-1)}}{e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(2a-1)}-e^{i\frac{2\pi}{2n+1}\frac{1}{2}(2a-1)}}+\frac{1-e^{-i\frac{2\pi}{2n+1}n(2a-1)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(2a-1)}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(2a-1)}}+2n\right] \\
= & \frac{1}{2n+1}\left[\frac{e^{i\frac{2\pi}{2n+1}n(2a-1)}-e^{-i\frac{2\pi}{2n+1}n(2a-1)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(2a-1)}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(2a-1)}}+2n\right] .
\end{aligned}
$$

Since $2\pi n/(2n+1)=\pi-\pi/(2n+1)$, $2a-1$ is an odd number, $e^{i\pi(2a-1)}=e^{-i\pi(2a-1)}=-1$ and hence the expression is zero.
(II) Next, when $a\neq b$, we have

$$
\begin{aligned}
& \frac{4}{2n+1}\sum_{j=1}^{n}\sin\frac{2\pi}{2n+1}a(j-\frac{1}{2})\sin\frac{2\pi}{2n+1}b(j-\frac{1}{2}) \\
= & \frac{1}{2n+1}(-1)\left[\sum_{j=n}^{n}\left(e^{i\frac{2\pi}{2n+1}a(j-\frac{1}{2})}-e^{-i\frac{2\pi}{2n+1}a(j-\frac{1}{2})}\right)\left(e^{i\frac{2\pi}{2n+1}b(j-\frac{1}{2})}-e^{-i\frac{2\pi}{2n+1}b(j-\frac{1}{2})}\right)\right] \\
= & \frac{-1}{2n+1}\sum_{j=1}^{n}\left[e^{i\frac{2\pi}{2n+1}(a+b)(j-\frac{1}{2})}+e^{-i\frac{2\pi}{2n+1}(a+b)(j-\frac{1}{2})}-e^{i\frac{2\pi}{2n+1}(a-b)(j-\frac{1}{2})}+e^{-i\frac{2\pi}{2n+1}(a-b)(j-\frac{1}{2})}\right] \\
= & \frac{-1}{2n+1}\left[e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b)}\frac{1-e^{i\frac{2\pi}{2n+1}n(a+b)}}{1-e^{i\frac{2\pi}{2n+1}(a+b)}}+e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b)}\frac{1-e^{-i\frac{2\pi}{2n+1}n(a+b)}}{1-e^{-i\frac{2\pi}{2n+1}(a+b)}}\right. \\
& \qquad \left.-\left[e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}\frac{1-e^{i\frac{2\pi}{2n+1}n(a-b)}}{1-e^{i\frac{2\pi}{2n+1}(a-b)}}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}\frac{1-e^{-i\frac{2\pi}{2n+1}n(a-b)}}{1-e^{-i\frac{2\pi}{2n+1}(a-b)}}\right]\right] \\
= & \frac{1}{2n+1}\left[\frac{1-e^{i\frac{2\pi}{2n+1}n(a+b)}}{e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b)}-e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}}+\frac{1-e^{-i\frac{2\pi}{2n+1}n(a-b)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b)}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}}\right. \\
& \qquad \left.-\frac{1-e^{i\frac{2\pi}{2n+1}n(a-b)}}{e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}-e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}}-\frac{1-e^{-i\frac{2\pi}{2n+1}n(a-b)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}}\right] \\
= & \frac{-1}{2n+1}\left[\frac{e^{i\frac{2\pi}{2n+1}n(a+b)}-e^{-i\frac{2\pi}{2n+1}n(a+b)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b)}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b)}}-\frac{e^{i\frac{2\pi}{2n+1}n(a-b)}-e^{-i\frac{2\pi}{2n+1}n(a-b)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}}\right] .
\end{aligned}
$$

Since $2\pi n/(2n+1) = \pi - \pi/(2n+1)$,

$$\frac{-1}{2n+1}\left[\frac{e^{i\pi(a+b)}e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b)} + e^{-i\pi(a+b)}e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b)} - e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b)}} - \frac{e^{i\pi(a-b)}e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)} - e^{-i\pi(a-b)}e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)} - e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}}\right].$$

When $a+b$ is odd, $a-b$ is odd. Then $e^{i\pi(a+b)} = e^{-i\pi(a+b-} = -1$, $e^{i\pi(a-b)} = e^{-i\pi(a-b)} = -1$ and hence the expression is zero. When $a+b$ is an even number, $a-b$ is even. Since $e^{i\pi(a+b)} = e^{-i\pi(a+b)} = 1$, $e^{i\pi(a-b)} = e^{-i\pi(a-b)} = 1$ and hence the expression is zero.
When $a-b=0$,

$$\begin{aligned}
& \frac{-1}{2n+1}[\sum_{j=n}^{n}\left(e^{i\frac{2\pi}{2n+1}(a+b)(j-\frac{1}{2})} - e^{i\frac{2\pi}{2n+1}(a-b)(j-\frac{1}{2})} + e^{-i\frac{2\pi}{2n+1}(a+b)(j-\frac{1}{2})} - e^{-i\frac{2\pi}{2n+1}(a-b)(j-\frac{1}{2})}\right)] \\
= & \frac{-1}{2n+1}\left[e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b)}\frac{1-e^{i\frac{2\pi}{2n+1}n(a+b)}}{1-e^{i\frac{2\pi}{2n+1}(a+b)}} + e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b)}\frac{1-e^{-i\frac{2\pi}{2n+1}n(a+b)}}{1-e^{-i\frac{2\pi}{2n+1}(a+b)}} - 2n\right] \\
= & \frac{-1}{2n+1}\left[\frac{1-e^{i\frac{2\pi}{2n+1}n(2a)}}{e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(2a)} - e^{i\frac{2\pi}{2n+1}\frac{1}{2}(2a-1)}} + \frac{1-e^{-i\frac{2\pi}{2n+1}n(2a)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(2a)} - e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(2a)}} - 2n\right] \\
= & \frac{-1}{2n+1}\left[\frac{e^{i\frac{2\pi}{2n+1}n(2a)} - e^{-i\frac{2\pi}{2n+1}n(2a)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(2a)} - e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(2a)}} - 2n\right].
\end{aligned}$$

Since $2\pi n/(2n+1) = \pi - \pi/(2n+1)$, $e^{i\pi(2a)} = e^{-i\pi(2a)} = 1$, we have 1.

(III) Third, when $a \neq b$, we have

$$
\begin{aligned}
& \frac{4}{2n+1}\sum_{j=1}^{n}\sin\frac{2\pi}{2n+1}(a-\frac{1}{2})j\sin\frac{2\pi}{2n+1}(b-\frac{1}{2})j \\
= \; & \frac{1}{2n+1}(-1)\left[\sum_{j=n}^{n}\left(e^{i\frac{2\pi}{2n+1}(a-\frac{1}{2})j}-e^{-i\frac{2\pi}{2n+1}(a-\frac{1}{2})j}\right)\left(e^{i\frac{2\pi}{2n+1}(b-\frac{1}{2})j}-e^{-i\frac{2\pi}{2n+1}(b-\frac{1}{2})j}\right)\right] \\
= \; & \frac{-1}{2n+1}\sum_{j=1}^{n}\Big]e^{i\frac{2\pi}{2n+1}(a+b-1)j}+e^{-i\frac{2\pi}{2n+1}(a+b-1)j}-e^{i\frac{2\pi}{2n+1}(a-b)j}+e^{-i\frac{2\pi}{2n+1}(a-b)j}\Big] \\
= \; & \frac{-1}{2n+1}\left[e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}\frac{1-e^{i\frac{2\pi}{2n+1}n(a+b-1)}}{1-e^{i\frac{2\pi}{2n+1}(a+b-1)}}+e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}\frac{1-e^{-i\frac{2\pi}{2n+1}n(a+b-1)}}{1-e^{-i\frac{2\pi}{2n+1}(a+b-1)}}\right. \\
& \qquad \left.-\big[e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}\frac{1-e^{i\frac{2\pi}{2n+1}n(a-b)}}{1-e^{i\frac{2\pi}{2n+1}(a-b)}}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}\frac{1-e^{-i\frac{2\pi}{2n+1}n(a-b)}}{1-e^{-i\frac{2\pi}{2n+1}(a-b)}}\right] \\
= \; & \frac{-1}{2n+1}\left[\frac{1-e^{i\frac{2\pi}{2n+1}n(a+b)}}{e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b)}-e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}}+\frac{1-e^{-i\frac{2\pi}{2n+1}n(a-b)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b)}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}}\right. \\
& \qquad \left.-\frac{1-e^{i\frac{2\pi}{2n+1}n(a-b)}}{e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}-e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}}-\frac{1-e^{-i\frac{2\pi}{2n+1}n(a-b)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}}\right] \\
= \; & \frac{-1}{2n+1}\left[\frac{e^{i\frac{2\pi}{2n+1}(a+b-1)\frac{1}{2}}-e^{-i\frac{2\pi}{2n+1}(n+\frac{1}{2})(a+b-1)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}-e^{-\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}}+\frac{e^{-i\frac{2\pi}{2n+1}(a+b-1)\frac{1}{2}}-e^{-i\frac{2\pi}{2n+1}(n+\frac{1}{2}(a+b-1)}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}-e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-1)}}\right. \\
& \qquad \left.-\frac{e^{i\frac{2\pi}{2n+1}(a-b)\frac{1}{2}}-e^{-i\frac{2\pi}{2n+1}(n+\frac{1}{2})(a-b)}}{e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}-e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}}-\frac{e^{-i\frac{2\pi}{2n+1}(a-b)\frac{1}{2}}-e^{-i\frac{2\pi}{2n+1}(n+\frac{1}{2})(a-b)}}{e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}-e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b)}}\right] .
\end{aligned}
$$

Therefore,

$$\frac{-1}{2n+1}\left[-1+\frac{2i\sin\pi(a+b-1)}{2i\sin\frac{2\pi}{2n+1}(a+b-1)\frac{1}{2}}+1-\frac{2i\sin\pi(a-b)}{2i\sin\frac{2\pi}{2n+1}(a-b)\frac{1}{2}}\right]$$

and hence the expression is zero. When $a-b=0$,

$$\frac{-1}{2n+1}\left[-1+\frac{2i\sin\pi(2a-1)}{2i\sin\frac{2\pi}{2n+1}(2a-1)\frac{1}{2}}-2n\right]$$

and we have 1.
Q.E.D.

Lemma A.2* (Orthogonality of the two discrete transforms) [3]

For $1 \le k, j \le n$, define

$$p_{kj}^{(n)} = \sqrt{\frac{4}{2n+1}} \cos\left(\frac{2\pi}{2n+1}(k-\tfrac{1}{2})(j-\tfrac{1}{2}) \right), \tag{1}$$

$$p_{kj}^{*(n)} = \sqrt{\frac{4}{2n+1}} \sin\left(\frac{2\pi}{2n+1}k(j-\tfrac{1}{2}) \right). \tag{2}$$

Let $P_n = (p_{kj}^{(n)})$ and $P_n^* = (p_{kj}^{*(n)})$. Then both matrices are real orthogonal:

$$P_n P_n' = P_n' P_n = I_n, \qquad P_n^* (P_n^*)' = (P_n^*)' P_n^* = I_n. \tag{3}$$

In particular, for all $a, b \in \{1, \dots, n\}$,

$$\sum_{j=1}^{n} p_{aj}^{(n)} p_{bj}^{(n)} = \sum_{j=1}^{n} p_{aj}^{*(n)} p_{bj}^{*(n)} = \sum_{j=1}^{n} p_{ja}^{*(n)} p_{jb}^{*(n)} = \delta_{ab}, \tag{4}$$

where $\delta_{ab} = 1$ if $a = b$ and 0 otherwise. Consequently, for each *fixed* $b \in \{1, \dots, n\}$,

$$\sum_{a=1}^{n}\sum_{j=1}^{n} p_{aj}^{(n)} p_{bj}^{(n)} = \sum_{a=1}^{n}\sum_{j=1}^{n} p_{aj}^{*(n)} p_{bj}^{*(n)} = \sum_{a=1}^{n}\sum_{j=1}^{n} p_{ja}^{*(n)} p_{jb}^{*(n)} = 1. \tag{5}$$

Preparatory calculations: every finite-sum identity used in the following proofs. The following elementary steps are included so that the proof can be reconstructed without consulting a table of trigonometric identities. Throughout this discussion let $N = 2n+1$, and put $\theta_j = 2\pi(j-\frac{1}{2})/N$. Notice that N is odd and that $1 \le j \le n$.

Step A: Why the product-to-sum identities hold. Starting from the angle-addition formulas,

$$\cos(x-y) = \cos x \cos y + \sin x \sin y, \qquad \cos(x+y) = \cos x \cos y - \sin x \sin y,$$

addition and subtraction give, respectively,

$$2\cos x \cos y = \cos(x-y) + \cos(x+y), \tag{6}$$

$$2\sin x \sin y = \cos(x-y) - \cos(x+y). \tag{7}$$

[3] The statements and its proofs of this proposition were suggested by ChatGPT in October 2026.

These are the only trigonometric product identities needed below.

Step B: Derivation of the finite cosine-sum formula. For real t with $\sin(t/2) \neq 0$, the elementary identity

$$2\sin(t/2)\cos((j-\tfrac{1}{2})t) = \sin(jt) - \sin((j-1)t)$$

follows from $2\sin u\cos v = \sin(u+v) + \sin(u-v)$. Summing this identity for $j = 1, \ldots, n$ makes the right-hand side *telescope*: all intermediate sine terms cancel, leaving

$$2\sin(t/2)\sum_{j=1}^{n}\cos((j-\tfrac{1}{2})t) = \sin(nt).$$

Consequently,

$$\boxed{\sum_{j=1}^{n}\cos((j-\tfrac{1}{2})t) = \frac{\sin(nt)}{2\sin(t/2)}.} \tag{8}$$

This formula is valid whenever $\sin(t/2) \neq 0$; at $t = 0$ the sum is simply n. To obtain the alternative form sometimes quoted as "the finite cosine-sum formula," use $\sin(2u) = 2\sin u\cos u$:

$$\frac{\sin(nt)}{2\sin(t/2)} = \frac{\sin(nt/2)}{\sin(t/2)}\cos(nt/2).$$

The two forms are identical, not independent assumptions.

Step C: Evaluation on the odd frequency grid. Set $t = 2\pi r/N$ for an integer r with $0 < |r| \leq 2n$. Since $N = 2n+1$, r is not a multiple of N; hence $\sin(\pi r/N) \neq 0$. Formula (8) yields

$$C_n(r) := \sum_{j=1}^{n}\cos(r\theta_j) = \frac{\sin(2\pi nr/N)}{2\sin(\pi r/N)} \tag{9}$$

$$= \frac{\sin(\pi r - \pi r/N)}{2\sin(\pi r/N)} = \frac{(-1)^{r+1}}{2}. \tag{10}$$

The last equality follows by expanding $\sin(\pi r - x) = \sin(\pi r)\cos x - \cos(\pi r)\sin x$ and using $\sin(\pi r) = 0$, $\cos(\pi r) = (-1)^r$. For $r = 0$, directly $C_n(0) = n$. Also $C_n(-r) = C_n(r)$, since cosine is even. Thus the exact rule used in both proofs is

$$C_n(r) = \begin{cases} n, & r = 0, \\ (-1)^{r+1}/2, & 0 < |r| \leq 2n. \end{cases} \tag{11}$$

For example, nonzero odd r give $C_n(r) = 1/2$, whereas nonzero even r give $C_n(r) = -1/2$. This parity distinction is the reason that the cross-products vanish.

Step D: The geometric series and the exponential form. For any complex $q \neq 1$, multiplication of $1 + q + \cdots + q^{n-1}$ by $1 - q$ cancels all interior terms:

$$(1 - q)(1 + q + \cdots + q^{n-1}) = 1 - q^n.$$

Thus $\sum_{j=0}^{n-1} q^j = (1 - q^n)/(1 - q)$. For $t = 2\pi r/N$ and $q = e^{it}$,

$$\sum_{j=1}^{n} e^{i(j-1/2)t} = e^{it/2} \frac{1 - e^{int}}{1 - e^{it}},$$
$$1 - e^{int} = -2ie^{int/2} \sin(nt/2),$$
$$1 - e^{it} = -2ie^{it/2} \sin(t/2).$$

Substitution, including cancellation of the factors $-2i$ and $e^{it/2}$, gives the exact complex identity

$$\boxed{\sum_{j=1}^{n} e^{i(j-1/2)t} = e^{int/2} \frac{\sin(nt/2)}{\sin(t/2)}.} \tag{12}$$

Taking real parts reproduces (8), because $\Re(e^{int/2}) = \cos(nt/2)$. At $r = 0$ we do *not* use the quotient: each summand is one, so the sum is n. For negative r the sum is the complex conjugate of the sum for positive r; their real parts agree.

Step E: Why orthonormal rows imply orthonormal columns. If a real $n \times n$ matrix A satisfies $AA' = I_n$, then $\det(A)^2 = \det(AA') = 1$, so A is invertible. Multiplying $AA' = I_n$ on the left by A^{-1} gives $A' = A^{-1}$ and hence $A'A = I_n$. This argument requires a *square* matrix; it is not a general assertion for rectangular matrices.

Proof 2 : Product-to-sum identities (the shorter derivation). Set $N = 2n + 1$ and $\theta_j = 2\pi(j - \frac{1}{2})/N$. Define

$$C_n(r) = \sum_{j=1}^{n} \cos(r\theta_j), \qquad r \in \mathbb{Z}.$$

By Steps B–C above, for $r \neq 0$ with $|r| \leq 2n$, we obtain

$$C_n(r) = \frac{\sin(n\pi r/N)}{\sin(\pi r/N)} \cos\left(\frac{n\pi r}{N}\right) \tag{13}$$
$$= \frac{\sin(2n\pi r/N)}{2\sin(\pi r/N)} = \frac{(-1)^{r+1}}{2}. \tag{14}$$

Indeed $2n = N - 1$ and $\sin(\pi r - \pi r/N) = (-1)^{r+1}\sin(\pi r/N)$. The denominator is nonzero in the stated range. Also $C_n(0) = n$. By (6),

$$\sum_{j=1}^{n} p_{aj}^{(n)} p_{bj}^{(n)} = \frac{2}{N}\{C_n(a-b) + C_n(a+b-1)\}. \tag{15}$$

If $a \neq b$, both arguments are nonzero and have opposite parity (since $(a + b - 1) - (a - b) = 2b - 1$ is odd); thus the two terms cancel. If $a = b$, the arguments are 0 and $2a - 1$, so the right-hand side is $(2/N)(n + \frac{1}{2}) = 1$. For the sine transform, the identity in (7) yields

$$\sum_{j=1}^{n} p_{aj}^{*(n)} p_{bj}^{*(n)} = \frac{2}{N}\{C_n(a-b) - C_n(a+b)\}. \tag{16}$$

If $a \neq b$, the nonzero integers $a - b$ and $a + b$ have the same parity (since their difference is $2b$), so the terms cancel. If $a = b$, the right-hand side is $(2/N)\{n-(-\frac{1}{2})\} = 1$ because $2a$ is even. We have proved $P_n P_n' = P_n^*(P_n^*)' = I_n$. Since both are square, row orthonormality implies invertibility and hence $P_n' P_n = (P_n^*)' P_n^* = I_n$. This proves the third equality in (4) without a separate column-sum calculation. Finally, summing δ_{ab} over a with b fixed proves (5). □

Proof 3: Complex exponentials and geometric series (a corrected version of the original approach). For any integer r, let

$$S_n(r) = \sum_{j=1}^{n} \exp\left(\frac{2\pi i r}{N}(j - \tfrac{1}{2})\right).$$

If $0 < |r| \leq 2n$, apply Step D and put $q = \exp(2\pi i r/N) \neq 1$. Then

$$S_n(r) = q^{1/2}\frac{1-q^n}{1-q} = \exp\left(\frac{\pi i r n}{N}\right)\frac{\sin(\pi r n/N)}{\sin(\pi r/N)}. \tag{17}$$

Here $q^{1/2}$ denotes specifically $\exp(\pi i r/N)$. Taking real parts gives

$$\Re S_n(r) = \frac{\sin(2\pi r n/N)}{2\sin(\pi r/N)} = \frac{(-1)^{r+1}}{2}, \tag{18}$$

while $S_n(0) = n$. This calculation supplies all finite-sum identities used below; in particular no geometric-series denominator is evaluated at $r = 0$.

(I) Cosine rows. Euler's formula implies

$$\begin{aligned}
&\sum_{j=1}^{n} p_{aj}^{(n)} p_{bj}^{(n)} \\
&= \frac{1}{N} \sum_{j=1}^{n} \left(e^{i(a-1/2)\theta_j} + e^{-i(a-1/2)\theta_j}\right)\left(e^{i(b-1/2)\theta_j} + e^{-i(b-1/2)\theta_j}\right) \\
&= \frac{1}{N}\{S_n(a+b-1) + S_n(a-b) + S_n(-(a-b)) + S_n(-(a+b-1))\} \\
&= \frac{2}{N}\{\Re S_n(a-b) + \Re S_n(a+b-1)\}.
\end{aligned}$$

For $a \neq b$, the two nonzero frequencies have opposite parity, so (18) gives zero. For $a = b$, $S_n(0) = n$ and $\Re S_n(2a-1) = 1/2$, giving one.
(II) Sine rows. Since $\sin x = (e^{ix} - e^{-ix})/(2i)$, careful multiplication gives

$$\begin{aligned}
&\sum_{j=1}^{n} p_{aj}^{*(n)} p_{bj}^{*(n)} \\
&= -\frac{1}{N} \sum_{j=1}^{n} (e^{ia\theta_j} - e^{-ia\theta_j})(e^{ib\theta_j} - e^{-ib\theta_j}) \\
&= \frac{1}{N}\{S_n(a-b) + S_n(-(a-b)) - S_n(a+b) - S_n(-(a+b))\} \\
&= \frac{2}{N}\{\Re S_n(a-b) - \Re S_n(a+b)\}.
\end{aligned}$$

For $a \neq b$, $a-b$ and $a+b$ are nonzero and of the same parity; therefore the result is zero. For $a = b$, $S_n(0) = n$ and $\Re S_n(2a) = -1/2$, giving one.
(III) Column orthogonality. From (I) and (II), $P_n P_n' = P_n^*(P_n^*)' = I_n$. For any square real matrix A, $AA' = I_n$ implies $A' = A^{-1}$, hence $A'A = I_n$. Applying this elementary fact to both matrices proves column orthogonality, including $\sum_j p_{ja}^{*(n)} p_{jb}^{*(n)} = \delta_{ab}$. The fixed-$b$ double sums follow as in Proof 1. □

Lemma A.3 : (i) From (43) in Sato and Kunitomo (SK (2026)), we find that

$$a_{n,m}(h) = \frac{1}{2}\left[1 - \frac{\sin 2\pi(h+1)\frac{2m}{2n+1}}{2m \sin 2\pi(h+1)\frac{1}{2n+1}}\right] . \tag{19}$$

(ii) From (46) in SK (2026), we find that

$$a_{n,m}^*(h) = [\frac{2m+1}{4m}]\left[1 - \frac{\sin 2\pi(h+1)\frac{2m+1}{2n+1}}{(2m+1) \sin 2\pi(h+1)\frac{1}{2n+1}}\right] . \tag{20}$$

Lemma A.4 : For any positive integers $a, b = 1, \cdots, m$ such that $m/n \to 0$ as $m, m \to \infty$,

$$\sum_{j=1}^{n} p_{aj}^{(n)} p_{bj}^{*(n)} = O(\frac{1}{n}) \ . \tag{21}$$

Proof of Lemma A.4 :
(I) Case $a \neq b$:

$$\begin{aligned}
& \frac{4}{2n+1} \sum_{j=1}^{n} \cos \frac{2\pi}{2n+1}(a-\frac{1}{2})(j-\frac{1}{2}) \sin \frac{2\pi}{2n+1} b(j-\frac{1}{2}) \\
= \ & \frac{1}{(2n+1)i} \sum_{j=1}^{n} \Big[\big(e^{i\frac{2\pi}{2n+1}(a-\frac{1}{2})(j-\frac{1}{2})} + e^{-i\frac{2\pi}{2n+1}(a-\frac{1}{2})(j-\frac{1}{2})}\big)\big(e^{i\frac{2\pi}{2n+1}b(j-\frac{1}{2})} - e^{-i\frac{2\pi}{2n+1}b(j-\frac{1}{2})}\big) \Big] \\
= \ & \frac{1}{(2n+1)i} \sum_{j=1}^{n} \Big[e^{i\frac{2\pi}{2n+1}(a+b-\frac{1}{2})(j-\frac{1}{2})} - e^{-i\frac{2\pi}{2n+1}(a+b-\frac{1}{2})(j-\frac{1}{2})} \\
& \qquad -\big[e^{i\frac{2\pi}{2n+1}(a-b-\frac{1}{2})(j-\frac{1}{2})} - e^{-i\frac{2\pi}{2n+1}(a-b-\frac{1}{2})(j-\frac{1}{2})}\big] \Big] \\
= \ & \frac{1}{2n+1} \left[e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-\frac{1}{2})} \frac{1-e^{i\frac{2\pi}{2n+1}n(a+b-\frac{1}{2})}}{1-e^{i\frac{2\pi}{2n+1}(a+b-\frac{1}{2})}} - e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-\frac{1}{2})} \frac{1-e^{-i\frac{2\pi}{2n+1}n(a+b-\frac{1}{2})}}{1-e^{-i\frac{2\pi}{2n+1}(a+b-\frac{1}{2})}} \right. \\
& \qquad \left. -\big[+\big[e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b-\frac{1}{2})} \frac{1-e^{i\frac{2\pi}{2n+1}n(a-b-\frac{1}{2})}}{1-e^{i\frac{2\pi}{2n+1}(a-b-\frac{1}{2})}} - +e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b-\frac{1}{2})} \frac{1-e^{-i\frac{2\pi}{2n+1}n(a-b-\frac{1}{2})}}{1-e^{-i\frac{2\pi}{2n+1}(a-b-\frac{1}{2})}}\big] \right] \\
= \ & \frac{1}{(2n+1)i} \left[\frac{1-e^{i\frac{2\pi}{2n+1}n(a+b-\frac{1}{2})}}{e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-\frac{1}{2})} - e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-\frac{1}{2})}} - \frac{1-e^{-i\frac{2\pi}{2n+1}n(a+b-\frac{1}{2})}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-\frac{1}{2})} - e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a+b-\frac{1}{2})}} \right. \\
& \qquad \left. -\big[\frac{1-e^{i\frac{2\pi}{2n+1}n(a-b-\frac{1}{2})}}{e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b-\frac{1}{2})} - e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b-\frac{1}{2})}} - \frac{1-e^{-i\frac{2\pi}{2n+1}n(a-b-\frac{1}{2})}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(a-b-\frac{1}{2})} - e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(a-b-\frac{1}{2})}}\big] \right] \\
= \ & \frac{1}{(2n+1)i} \left[\frac{-1+\cos\pi\frac{a+b-\frac{1}{2}}{2n+1}}{i\sin 2\pi\frac{a+b-\frac{1}{2}}{2n+1}\frac{1}{2}} - \frac{-1+\cos\pi\frac{a-b-\frac{1}{2}}{2n+1}}{i\sin 2\pi\frac{a-b-\frac{1}{2}}{2n+1}\frac{1}{2}} \right] .
\end{aligned}$$

Here we use the half-angle identity ($2\sin^2(x/2) = 1-\cos x$, $(\sin x = 2\sin(x/2)\cos(x/2)$), yielding the right-hand side

$$\frac{1}{2n+1} \left[\frac{\sin\pi\frac{a+b-\frac{1}{2}}{2n+1}\frac{1}{2}}{2\cos\pi\frac{a+b-\frac{1}{2}}{2n+1}\frac{1}{2}} - \frac{\sin\pi\frac{a-b-\frac{1}{2}}{2n+1}\frac{1}{2}}{2\cos\pi\frac{a-b-\frac{1}{2}}{2n+1}\frac{1}{2}} \right]$$

This follows. Furthermore, if $a, b = 1, \cdots, n$, $0 < \pi\frac{a+b-\frac{1}{2}}{2n+2}\frac{1}{2} < \pi/2$, $0 < |\pi\frac{a-b-\frac{1}{2}}{2n+2}\frac{1}{2}| < \pi/2$ hence $0 < \cos\pi\frac{a+b-\frac{1}{2}}{2n+2}\frac{1}{2} \leq 1$, $0 < |\cos\pi\frac{a-b-\frac{1}{2}}{2n+2}\frac{1}{2}| \leq$ hence the left-hand side is $O(1/n)$.
(II) Case $a = b$: In (I), the third and fourth terms on the third line cancel each other, so the claim follows from the first and second terms.

Remark 1 : In the previous versions of the paper, we mentioned to the statistic $d_{n-h} = W_{f,n} - W_{b,n}$ and its asymptotic distribution. However, we found that the following result holds only when $a, b = 1, \cdots, m$ and $m/n \to 0$ and its asymptotic variance becomes more complicated than we had explained in the previous versions. Thus we omit the discussions on this statistic in the present version.

On Proof of Theorem 5.5 :

For the backward smoothong, we use a similar evaluation as Theorem 5.4. We consider the general stationary processes when $\Delta x_t = x_t - x_{t-1} = v_t^{(x)}$ and v_t $(t = 1, \cdots, n)$ follow (1)-(4) in SK (2026), and we set the initial conditions $x_n = v_n = 0$ for the convenience and take h as a fixed positive integer. We utilize the representation of x_{n-h} in (1)-(4) SK (2026), $\Delta^{(b)}x_{n-h} = x_{n-h}^{(b)} - x_{n-h-1}^{(b)} = \mathbf{e}_{n-h}^{'}\mathbf{P}_n^{*'}\mathbf{Q}_n^{(m)}\mathbf{P}_n^{*}\mathbf{C}_n^{'-1}(\mathbf{Y}_n^* - \bar{\mathbf{Y}}_n^*)$ and $\mathbf{C}_n^{-1}(\mathbf{Y}_n - \bar{\mathbf{Y}}_0) = -(v_t^{(x)} + \Delta v_t)$ is an $n \times 1$ vector when $p = 1$. Then we need to evaluate the effects of transformation $\mathbf{P}_n^{*'}\mathbf{Q}_n^{(m)}\mathbf{P}_n^*$ without the factor $2/(n + 1/2)$.

We use (1)-(4) in SK (2026) for the general case when $p = 1$, and we set

$$r_t = \Delta x_t = \sum_{s=-\infty}^{\infty} C_s^{(x)} e_{t-s}^{(x)} , \tag{22}$$

where (a) $\sum_{s=-\infty}^{\infty} |C_s^{(x)}| < \infty$ and (b) $\sup \int_{|e_t^{(x)}|>c} v^2 dF(e) \longrightarrow 0$ $(c \longrightarrow \infty)$ (F is the distribution function of $e_s^{(x)}$).
Let $r_{t,k} = \sum_{s=-k}^{k} C_s^{(x)} e_{t-s}^{(x)}$. Then, by using the standard method of Anderson (1971, Section 7.7 and Page 430), we can apply the CLT to the truncated sum to obtain the asymptotic normality. It is because $\mathbf{E}[r_t - r_{t,k}]^2 \leq (\sum_{|s|>k} |C_s^{(x)}|)^2 \mathbf{Var}(e_t^{(x)})$ and the difference of r_t and $r_{t,k}$, the truncated sum, is stochastically negligible as $k \to \infty$ by the condition $\sum_{s=-\infty}^{\infty} |C_s^{(x)}| < +\infty$.)
For a fixed non-negative integer h and a positive integer k, we decompose

$R_{n-h} = R_{1,n-h} + R_{2,n-h}$ $(n = 1, \cdots, n)$, and define

$$\begin{aligned}
R_{1,n-h} &= a_n \sum_{t=1}^{n} \sum_{j=1}^{m} A^*(n-h,t,j) r_{t,k} \\
&= a_n \sum_{t=1}^{n} \sum_{s=-k}^{k} C_s^{(x)} \sum_{j=1}^{m} A^*(n-h,t,j) e_{t-s}^{(x)} \\
&= a_n \sum_{t'=k+1}^{n-k} \sum_{s=-k}^{k} C_s^{(x)} \sum_{j=1}^{m} A^*(n-h,t^{'}+s,j) e_{t'}^{(x)} \\
&\qquad + a_n \sum_{t'=1-k}^{k} \sum_{s=-t'+1}^{k} C_s^{(x)} \sum_{j=1}^{m} A^*(n-h,t^{'}+s,j) e_{t'}^{(x)} \\
&\qquad + a_n \sum_{t'=n-k+1}^{n+k} \sum_{s=-k}^{n-t^{'}} C_s^{(x)} \sum_{j=1}^{m} A^*(n-h,t^{'}+s,j) e_{t'}^{(x)} \\
&= W_{b,1n} + W_{b,2n} + W_{b,3n} \ \ \text{(say)} \ ,
\end{aligned}$$

where $A^*(n-h,t^{'},j) = \sin\frac{2\pi}{2n+1}(n-h-\frac{1}{2})j \sin\frac{2\pi}{2n+1}j(t^{'}-\frac{1}{2})$ and $R_{2,n-h} = \sum_{t=1}^{n}\sum_{j=1}^{m} A^*(n-h,t,j)\Delta v_t$.

Although we need several steps to obtain the results, we only give an outline of our arguments. The evaluation of the effect of truncations and derivations could be lengthy, but its most parts are straightforward extensions of those of Theorems 5.1 and 5.3.

(i) To show the asymptotic distribution of R_{n-h}, we first decompose $W_{b,1n}$ in $R_{1,n-h}$ as $\sum_{t'=k+1}^{n-k}[\,\cdot\,] = \sum_{t'=1}^{n}[\,\cdot\,] - \sum_{t'=1}^{k}[\,\cdot\,] - \sum_{t'=n-k+1}^{n}[\,\cdot\,]$. We apply the CLT to the first term and we can obtain the asymptotic normality as Theorem 5.3 with the additional variance term $(\sum_{s=-k}^{k} C_s^{(x)})^2$. It is because by using the relation $\sin\frac{2\pi}{2n+1}(n-h-\frac{1}{2})j = \sin\pi j \sin\pi(h+1)\frac{2j}{2n+1}$ in the proof of Theorem 5.3, the variance of the first term becomes approximately as $m/n \to 0$,

$$\begin{aligned}
\mathbf{Var}[R_{1,n-h}] &= (\sum_{s=-k}^{k} C_s^{(x)})^2 \sigma_e^{(x)2} a_n^2 [\frac{n}{2} + \frac{1}{4}] \sum_{j=1}^{m} [\sin\frac{2\pi}{2n+1}(n-h-\frac{1}{2})j]^2 \\
&\sim (\sum_{s=-k}^{k} C_s^{(x)})^2 \sigma_e^{(x)2} [\frac{2m}{n}] a_{n,m}^*(h) \ ,
\end{aligned}$$

where we denote $\mathbf{E}[e_s^{(x)2}] = \sigma_e^{(x)2}$.

Here wee need to show that the second and third terms of $W_{1,n-h}$ are stochastically negligible. It is a typical argument and we shall show it in the next

step.
(ii) The second and third terms in $\sum_{t'=1}^{k}[\ \cdot\ ]$ and $\sum_{t'=n-k+1}^{n}[\ \cdot\ ]$ for $W_{f,1n}$ are stochastically negligible. First, we notice that $|A^*(n-h,t^{'}+s,j)-A^*(n-h,t^{'},j)|=O(m/n)$ for fixed k and h. The first term is $\sum_{t'=1}^{k}[\ \cdot\ ]$ and, given k, its variance is less than

$$\begin{aligned} & K_1 \times \mathbf{Var}[(\sum_{s=-k}^{k} C_s) a_n \sum_{j=1}^{m} (A(n-h,t^{'},j)+o(1))] \\ \leq\ & K_1 (\sum_{s=-k}^{k} C_s)^2 a_n^2 [\sum_{j=1}^{m} (\sin \frac{2\pi}{2n+1}(n-h-\frac{1}{2})j + o(1)]^2 \end{aligned}$$

where K_1 is a positive constant. It is less than $K_1^{'} \times [\frac{2m}{n+\frac{1}{2}}] a_{n,m}^*(h)$ and $K_1^{'}$ is a positive constant.
The second term is $\sum_{t'=n-k+1}^{n}[\ \cdot\ ]$ and given k, its variance is less than $K_2 \times [\frac{2m}{(n+\frac{1}{2})^2}] a_{n,m}^*(h)$ by using the same relation and K_2 is a positive constant.
(iii) We evaluate $W_{f,2n}$ and $W_{f,3n}$, which are are stochastically negligible by using similar evaluations. It is because the number of each terms are $2k$ and we can apply the same evaluation method as (ii).
(iv) For the term $R_{2,n-h}=a_n \sum_{t=1}^{n}\sum_{j=1}^{m} A^*(n-h,t,j)\Delta v_t$, we approximate $\Delta v_t = \sum_{s=-\infty}^{\infty} C_s^{(v)} e_{t-s}^{(v)}$ by $\Delta v_{t,k} = \sum_{s=-k}^{k} C_s^{(v)} e_{t-s}^{(v)}$ for a positive integer k, which is constructed as $R_{1,n-h}$ by replacing $e_{t-s}^{(x)}$ by $\Delta e_{t-s}^{(v)}$. We decompose $R_{2,n-h}$ as

$$\begin{aligned} R_{2,n-h} &= a_n \sum_{t=1}^{n}\sum_{j=1}^{m} A^*(n-h,t,j)\Delta v_{t,k} \\ &= a_n \sum_{t=1}^{n}\sum_{s=-k}^{k} C_s^{(v)} \sum_{j=1}^{m} A^*(n-h,t,j)\Delta e_{t-s}^{(v)} \\ &= a_n \sum_{t'=k+1}^{n-k}\sum_{s=-k}^{k} C_s^{(v)} \sum_{j=1}^{m} A^*(n-h,t^{'}+s,j)\Delta e_{t^{'}}^{(v)} \\ &\qquad + a_n \sum_{t'=1-k}^{k}\sum_{s=-t'+1}^{k} C_s^{(v)} \sum_{j=1}^{m} A^*(n-h,t^{'}+s,j)\Delta e_{t^{'}}^{(v)} \\ &\qquad + a_n \sum_{t'=n-k+1}^{n+k}\sum_{s=-k}^{n-t^{'}} C_s^{(v)} \sum_{j=1}^{m} A^*(n-h,t^{'}+s,j)\Delta e_{t^{'}}^{(v)} \\ &= S_{f,1n} + S_{f,2n} + S_{f,3n} \ \ (\text{say})\ , \end{aligned}$$

Then, we need to show that all terms of $R_{2,n-h}$ are stochastically negligible because of the evaluation of their variances.
We first decompose $S_{f,1n}$ in $S_{1,n-h}$ as $\sum_{t'=k+1}^{n-k}[\ \cdot\ ] = \sum_{t'=1}^{n}[\ \cdot\ ] - \sum_{t'=1}^{k}[\ \cdot\ ] - \sum_{t'=n-k+1}^{n}[\ \cdot\ ]$. Then we can find that the first term is $\sum_{s=-k}^{k} C_s^{(v)} U_{2,n-h}$ plus terms of smaller orders, where $U_{2,n-h}$ is defined in Theorem 5.1.
Since $U_{2,n-h} = O(\frac{m}{n^2} a_{n,m}(h))$, its stochastic order is smaller than the corresponding term of $R_{1,n-h}$.
By using the similar evaluations for $W_{1,n-h}$, the second and third terms of $S_{1,n-h}$ are stochastically negligible. Also, $S_{2,n-h}$ and $S_{3,n-h}$ are stochastically negligible.
Hence,, by evaluating $\mathbf{Var}[R_{2,n-h}]$, which is associated with terms $\Delta v_t^{(v)}$, as in Theorems 5.1 and Theorem 5.3, it is straightforward to show that $R_{2,n-h}$ is asymptotically negligible when $n \to \infty$ while $m/n \to 0$ in comparison with $\mathbf{Var}[R_{1,n-h}]$.
(v) Finally, because the spectral density at zero is given by
$f_{\Delta x}(0) = (\sum_{s=-\infty}^{\infty} C_s^{(x)})^2 \sigma_e^{(x)2}$, we have the result by taking $k \to \infty$. Consistency of $\hat{\sigma}_x^2$ was proven in Theorem 5.2 of SK (2026) in the general case. Then, we have the result.
(**Q.E.D**)

Lemma A.5 : For any positive integers $a \leq n$, $m \leq n$,

$$\sum_{j=1}^{m} \cos \frac{2\pi}{2n+1}(j - \frac{1}{2})a = \frac{1}{2} \frac{\sin \frac{2\pi}{2n+1} m\ a}{\sin \frac{\pi}{2n+1} a}\ , \tag{23}$$

and

$$\sum_{j=1}^{m} \sin \frac{2\pi}{2n+1}(j - \frac{1}{2})a = \frac{\sin^2 \frac{\pi}{2n+1} m\ a}{\sin \frac{\pi}{2n+1} a}\ . \tag{24}$$

Proof of Lemma A.5 :
(i) The first equality follows from Lemma 5.1 of Kunitomo–Sato–Kurisu (2017).

(ii) We prove the second equality.

$$\begin{aligned}
\sum_{j=1}^{m} \sin \frac{2\pi}{2n+1}(j-\frac{1}{2})a &= \frac{1}{2i}\sum_{j=1}^{m}\left[e^{i\frac{2\pi}{2n+1}a(j-\frac{1}{2})} + e^{-i\frac{2\pi}{2n+1}a(j-\frac{1}{2})}\right] \\
&= \frac{1}{2i}\left[e^{i\frac{2\pi}{2n+1}\frac{1}{2}a}\frac{1-e^{i\frac{2\pi}{2n+1}ma}}{1-e^{i\frac{2\pi}{2n+1}a}} - e^{-i\frac{2\pi}{2n+1}\frac{1}{2}a}\frac{1-e^{-i\frac{2\pi}{2n+1}ma}}{1-e^{-i\frac{2\pi}{2n+1}a}}\right] \\
&= \frac{1}{2i}\left[\frac{1-e^{i\frac{2\pi}{2n+1}ma}}{e^{-i\frac{2\pi}{2n+1}\frac{1}{2}a} - e^{i\frac{2\pi}{2n+1}\frac{1}{2}a}} - \frac{1-e^{-i\frac{2\pi}{2n+1}ma}}{e^{i\frac{2\pi}{2n+1}\frac{1}{2}a} - e^{-i\frac{2\pi}{2n+1}\frac{1}{2}a}}\right] \\
&= \frac{1}{2i}\left[\frac{-2+2\cos\frac{2\pi}{2n+1}ma}{2i\sin\frac{2\pi}{2n+1}\frac{1}{2}a}\right] \\
&= \frac{1}{2}\left[\frac{1-\cos\frac{2\pi}{2n+1}ma}{\sin\frac{2\pi}{2n+1}\frac{1}{2}a}\right] \\
&= \frac{\sin^2\frac{2\pi}{2n+1}\frac{1}{2}ma}{\sin\frac{2\pi}{2n+1}\frac{1}{2}a} \ .
\end{aligned}$$

Here we use the half-angle identity ($2\sin^2(x/2) = 1-\cos x$).
(**Q.E.D**)

On Proof of Part II of Theorem 3.1 :

(**Part II**) We can apply the similar arguments to T_{2k} ($k \geq 1$). By using the recursive relations, for $k \geq 1$ we can represent

$$T_{2k} = A_{1*}^{(m,n)} + A_{2*}^{(n)} T_{2(k-1)+1} \ , \tag{25}$$

where $A_{1*}^{(m,n)}$ and $A_{2*}^{(m,n)}$ are $n \times n$ matrices as defined in Theorem 3.1. By evaluating the eigenvalues of $A_{2*}^{(m,n)}$, we find that the absolute value of eigenvalues are less than one, and we have convergence of T_{2k} as $k \to \infty$.

We give some details of derivations. We consider the case of $T_{2(k+1)}$ ($k \geq 0$). By using the recursive relations, for $k \geq 0$ we represent

$$T_{2k+1} = A_{1}^{(m,n)} + A_{2}^{(m,n)} T_{2(k-1)+1} \ , \tag{26}$$

where an $n \times n$ matrix $A_{2}^{(m,n)}$ is defined by

$$A_{2*}^{(m,n)} = (\mathbf{I}_n - \mathbf{C}_n^{'}\mathbf{P}_n^{*'}\mathbf{J}_m^{'}\mathbf{J}_m\mathbf{P}_n^{*}\mathbf{C}_n^{-1'})\mathbf{1}_n\mathbf{e}_n^{'}(\mathbf{I}_n - \mathbf{C}_n\mathbf{P}_n\mathbf{J}_m^{'}\mathbf{J}_m\mathbf{P}_n\mathbf{C}_n^{-1})\mathbf{1}_n\mathbf{e}_1^{'} \ . \tag{27}$$

Then, we consider the characteristic roots of the coefficient matrix $\mathbf{A}_{2*}^{(m,n)}$. Because the rank of $\mathbf{A}_{2*}^{(m,n)}$ is one, there are $n-1$ zero roots and one non-zero root, which is

$$\begin{aligned} a_{2n}^{*} &= \mathbf{e}_1^{'}(\mathbf{I}_n - \mathbf{C}_n^{'}\mathbf{P}_n^{*'}\mathbf{J}_m^{'}\mathbf{J}_m\mathbf{P}_n^{*}\mathbf{C}_n^{-1'})\mathbf{1}_n\mathbf{e}_n^{'}(\mathbf{I}_n - \mathbf{C}_n\mathbf{P}_n\mathbf{J}_m^{'}\mathbf{J}_m\mathbf{P}_n\mathbf{C}_n^{-1})\mathbf{1}_n \\ &= [1 - \mathbf{1}_n^{`}\mathbf{P}_n^{*'}\mathbf{J}_m^{'}\mathbf{J}_m\mathbf{P}_n^{*}\mathbf{e}_n][1 - \mathbf{1}_n^{`}\mathbf{P}_n\mathbf{J}_m^{'}\mathbf{J}_m\mathbf{P}_n\mathbf{e}_1] \ . \end{aligned} \tag{28}$$

In the above we used the relations as $\mathbf{e}_n^{'}\mathbf{C}_n = \mathbf{1}_n^{`}$ and $\mathbf{e}_1^{'}\mathbf{C}_n^{'} = \mathbf{1}_n^{`}$.
Then, we have the key relation

$$a_{2n}^{*} = a_{2n} \tag{29}$$

and we have the result.

(**Remark 2**) : We have an alternative evaluation. The first term of (28) becomes

$$[\sqrt{\frac{2}{n+\frac{1}{2}}}]^2 \sum_{k=m+1}^{n} [\sum_{j=1}^{n} \sin\frac{2\pi}{2n+1}(j-\frac{1}{2})k] \times \sin\frac{2\pi}{2n+1}k(n-\frac{1}{2})] \ ,$$

which is less than 1.
By using the relation

$$\begin{aligned} \sin\frac{2\pi}{2n+1}k(n-\frac{1}{2}) &= \sin[\pi k - \pi\frac{2k}{2n+1}] = (-1)^{k+1}\sin\pi\frac{2k}{2n+1} \\ &= (-1)^{k+1}2\sin\pi\frac{k}{2n+1}\cos\pi\frac{k}{2n+1} \ . \end{aligned}$$

We use the elementary relation that

$$\sin^2\frac{2\pi}{2n+1}\frac{kn}{2} = [\sin^2\pi k\frac{2n+1-1}{2n+1}\frac{1}{2}]^2 = [\sin\frac{\pi k}{2}\cos\frac{\pi k}{2n+1}\frac{1}{2} - \cos\frac{\pi k}{2}\sin\frac{\pi k}{2n+1}\frac{1}{2}]^2$$

and Lemma 4,

$$\begin{aligned} & [\frac{2}{2n+1}]\sum_{k=m+1}^{n}[\frac{\sin^2\frac{2\pi}{2n+1}kn\frac{1}{2}}{\sin\frac{\pi}{2n+1}k}] \times \sin\frac{2\pi}{2n+1}k(n-\frac{1}{2}) \\ = \ & [\frac{4}{2n+1}]\sum_{k=m+1}^{n}(-1)^{k+1}\sin^2\frac{2\pi}{2n+1}kn\frac{1}{2}\cos\frac{\pi}{2n+1}k \ . \end{aligned}$$

We use the elementary relation that

$$\sin^2\frac{2\pi}{2n+1}\frac{kn}{2} = [\sin\pi k\frac{2n+1-1}{2n+1}\frac{1}{2}]^2 = [\sin\frac{\pi k}{2}\cos\frac{\pi k}{2n+1}\frac{1}{2} - \cos\frac{\pi k}{2}\sin\frac{\pi k}{2n+1}\frac{1}{2}]^2.$$

For an odd number of k, it is $[\cos\frac{\pi}{2}\frac{k}{2n+1}]^2$ while for an even number of k, it is $[\sin\frac{\pi}{2}\frac{k}{2n+1}]^2$.
By using $[\sin\frac{\pi}{2}\frac{k}{2n+1}]^2 = 1-[\cos\frac{\pi}{2}\frac{k}{2n+1}]^2$,

$$\begin{aligned}
& [\frac{4}{2n+1}]\sum_{k=m+1}^{n}(-1)^{k+1}\sin^2\frac{2\pi}{2n+1}kn\frac{1}{2}\cos\frac{\pi}{2n+1}k \\
< & [\frac{4}{2n+1}]\sum_{k=m+1}^{n}[\cos\frac{\pi}{2}\frac{k}{2n+1}]^2\cos\frac{\pi}{2n+1}k \\
< & [\frac{4}{2n+1}][\cos\frac{\pi}{2}\frac{m+1}{2n+1}]^2\sum_{k=m+1}^{n}\cos\frac{\pi}{2n+1}k \\
\leq & [\frac{4}{2n+1}][\frac{1}{\sqrt{2}}]^2(n-m) < 1\ .
\end{aligned}$$

Similarly, for the lower bound, we have

$$\begin{aligned}
& [\frac{4}{2n+1}]\sum_{k=m+1}^{n}(-1)^{k+1}\sin^2\frac{2\pi}{2n+1}kn\frac{1}{2}\cos\frac{\pi}{2n+1}k \\
> & [\frac{4}{2n+1}]\sum_{k=m+1}^{n}(-1)[\sin\frac{\pi}{2}\frac{k}{2n+1}]^2\cos\frac{\pi}{2n+1}k \\
\geq & [-\frac{4}{2n+1}][\sin\frac{\pi}{2}\frac{n}{2n+1}]^2\sum_{k=m+1}^{n}\cos\frac{\pi}{2n+1}k \\
\geq & [-\frac{4}{2n+1}](\frac{1}{\sqrt{2}})^2(n-m) > -1
\end{aligned}$$

because $0 < (\pi/2)k/(2n+1) < \pi/4$ and $0 < \sin(\pi/2)k/(2n+1) < 1/\sqrt{2}$.
Then, for any $n \geq 1$, $\mathbf{1}_n^{'}\mathbf{P}_n^{*'}\mathbf{J}_{n-m}^{'}\mathbf{J}_{n-m}\mathbf{P}_n^{*}\mathbf{e}_n$ is less than one.
Similarly, for the second term of (28), it is s less than 1 and we have $|a_{2n}^{*}| < 1$.
Then, we have convergence of T_{2k} as $k \to \infty$.
(**Q.E.D.**)

On Proof of Theorem 3.3 :

The proof is basically the same as Theorem 3.1. We replace $\mathbf{Q}_n^{(m)} = \mathbf{J}_m^{'}\mathbf{J}_m$ by $\mathbf{Q}_n^{(m_1,m_2)} = \mathbf{J}_{m_1,m_2}^{'}\mathbf{J}_{m_1,m_2}$.
Consider $T_{2k+1} = A_1^{(m_1,m_2,,n)} + A_2^{(m_1,m_2,n)}T_{2k-1}$. Then, the non-zero eigenvalue of $A_2^{(m_1,m_2,n)}$ is

$$\begin{aligned}
a_{2n} &= \mathbf{e}_n^{'}(\mathbf{I}_n - \mathbf{C}_n\mathbf{P}_n\mathbf{J}_{m_1,m_2}^{'}\mathbf{J}_{m_1,m_2}\mathbf{P}_n\mathbf{C}_n^{-1})\mathbf{1}_n\mathbf{e}_1^{'}(\mathbf{I}_n - \mathbf{C}_n^{'}\mathbf{P}_n^{*'}\mathbf{J}_{m_1,m_2}^{'}\mathbf{J}_{m_1,m_2}\mathbf{P}_n^{*}\mathbf{C}_n^{'-1})\mathbf{1}_n \\
&= [1 - \mathbf{1}_n^{'}\mathbf{P}_n\mathbf{J}_{m_1,m_2}^{'}\mathbf{J}_{m_1,m_2}\mathbf{P}_n\mathbf{e}_1][1 - \mathbf{1}_n^{'}\mathbf{P}_n^{*}\mathbf{J}_{m_1,m_2}^{'}\mathbf{J}_{m_1,m_2}\mathbf{P}_n^{*}\mathbf{e}_n]\ .
\end{aligned}$$

In this case, we use the relation

$$\begin{aligned}[1-\mathbf{1}_n^{'}\mathbf{P}_n\mathbf{J}_{m_1,m_2}^{'}\mathbf{J}_{m_1,m_2}\mathbf{P}_n\mathbf{e}_1] &= \mathbf{1}_n^{'}\mathbf{P}_n\mathbf{J}_{m_1}^{'}\mathbf{J}_{m_1}\mathbf{P}_n\mathbf{e}_1 \\ &+ \mathbf{1}_n^{'}\mathbf{P}_n\mathbf{J}_{m_1+m_2+1,n}^{'}\mathbf{J}_{m_1+m_2+1,n}\mathbf{P}_n\mathbf{e}_1\end{aligned}$$

and

$$\begin{aligned}[1-\mathbf{1}_n^{'}\mathbf{P}_n^{*'}\mathbf{J}_{m_1,m_2}^{'}\mathbf{J}_{m_1,m_2}\mathbf{P}_n^{*}\mathbf{e}_n] &= \mathbf{1}_n^{'}\mathbf{P}_n^{*'}\mathbf{J}_{m_1}^{'}\mathbf{J}_{m_1}\mathbf{P}_n^{*}\mathbf{e}_n \\ &+ \mathbf{1}_n^{'}\mathbf{P}_n^{*'}\mathbf{J}_{m_1+m_2+1,n}^{'}\mathbf{J}_{m_1+m_2+1,n}\mathbf{P}_n^{*}\mathbf{e}_n\ .\end{aligned}$$

Then, by using the same argument as the proof of Theorem 3.1 and Lemma A.2, we find that the absolute values of the eigenvalues of $\mathbf{A}_2^{(m_1,m_2,n}$ and $\mathbf{A}_{2*}^{(m_1,m_2,n}$ are less than one when m_1 and m_2 are even numbers.
(**Part I**) We consider the case of T_{2k+1} $(k \geq 1)$. By using the recursive relations, for $k \geq 1$ we represent

$$T_{2k+1} = A_1^{(m_1,m_2)} + A_2^{(m_1,m_2)}T_{2(k-1)+1}\ , \tag{30}$$

where an $n \times n$ matrix $A_2^{(m,n)}$ is defined by

$$A_2^{(m_1,m_2)} = (\mathbf{I}_n - \mathbf{C}_n\mathbf{P}_n\mathbf{Q}_n^{(m_1,m_2)}\mathbf{P}_n\mathbf{C}_n^{-1})\mathbf{1}_n\mathbf{e}_1^{'}(\mathbf{I}_n - \mathbf{C}_n^{'}\mathbf{P}_n^{*'}\mathbf{Q}_n^{(m_1,m_2)}\mathbf{P}_n^{*}\mathbf{C}_n^{'-1})\mathbf{1}_n\mathbf{e}_n^{'}\ . \tag{31}$$

Then, we consider the characteristic roots of the coefficient matrix $\mathbf{A}_2^{(m_1,m_2)}$. Because the rank of $\mathbf{A}_2^{(m_1,m_2)}$ is one, there are $n-1$ zero roots and one non-zero root, which is

$$\begin{aligned}a_{2n} &= \mathbf{e}_n^{'}(\mathbf{I}_n - \mathbf{C}_n\mathbf{P}_n\mathbf{Q}_n^{(m_1,m_2)}\mathbf{P}_n\mathbf{C}_n^{-1})\mathbf{1}_n\mathbf{e}_1^{'}(\mathbf{I}_n - \mathbf{C}_n^{'}\mathbf{P}_n^{*'}\mathbf{Q}_n^{(m_1,m_2)}\mathbf{P}_n^{*}\mathbf{C}_n^{'-1})\mathbf{1}_n \\ &= [1 - \mathbf{1}_n^{`}\mathbf{P}_n\mathbf{Q}_n^{(m_1,m_2)}\mathbf{P}_n\mathbf{e}_1][1 - \mathbf{1}_n^{`}\mathbf{P}_n^{*'}\mathbf{Q}_n^{(m_1,m_2)}\mathbf{P}_n^{*}\mathbf{e}_n]\ .\end{aligned} \tag{32}$$

We have used the relations as $\mathbf{e}_n^{'}\mathbf{C}_n = \mathbf{1}_n^{`}$ and $\mathbf{e}_1^{'}\mathbf{C}_n^{'} = \mathbf{1}_n^{`}$.
We evaluate two terms in (32). The first term of (32) becomes

$$[\sqrt{\frac{2}{n+\frac{1}{2}}}]^2 \sum_{k\in\mathbf{K}_n}[\sum_{j=1}^{n}\cos\frac{2\pi}{2n+1}(j-\frac{1}{2})(k-\frac{1}{2})] \times \cos\frac{2\pi}{2n+1}(k-\frac{1}{2})(1-\frac{1}{2})]\ ,$$

which is less than 1 for $\mathbf{K}_n = \{1, \cdots, m_1, m_1+m_2+1, \cdots, n\}$. We show this key fact in the following. We take the summation with respect to j as

$$[\frac{2}{2n+1}]\sum_{k\in\mathbf{K}_n}[[\frac{\sin\frac{2\pi}{2n+1}(k-\frac{1}{2})n}{\sin\frac{\pi}{2n+1}(k-\frac{1}{2})}] \times \cos\frac{2\pi}{2n+1}(k-\frac{1}{2})(1-\frac{1}{2})]$$

and the relation

$$\sin\frac{\pi}{2n+1}(k-\frac{1}{2})[2n+1-1] = \sin\pi(k-\frac{1}{2})\cos\frac{\pi}{2n+1}(k-\frac{1}{2}) \ .$$

Then the first term of (32) becomes

$$[\frac{2}{2n+1}]\sum_{k\in\mathbf{K}_n}\sin\pi(k-\frac{1}{2})\times[\frac{[\cos\frac{\pi}{2n+1}(k-\frac{1}{2})]^2}{\sin\frac{\pi}{2n+1}(k-\frac{1}{2})}] \ .$$

We use the facts that (i) $\sin\pi(k-\frac{1}{2})$ takes $+1$ and -1 alternatively, (ii) the absolute value of $\frac{[\cos\frac{\pi}{2n+1}(k-\frac{1}{2})]^2}{\sin\frac{\pi}{2n+1}(k-\frac{1}{2})}$ is a monotone decreasing for $k=1,\cdots,n$ and (iii)

$$[\sqrt{\frac{2}{n+\frac{1}{2}}}]^2\sum_{k=1}^{n}[\sum_{j=1}^{n}\cos\frac{2\pi}{2n+1}(j-\frac{1}{2})(k-\frac{1}{2})]\times\cos\frac{2\pi}{2n+1}(k-\frac{1}{2})(1-\frac{1}{2})] = 1 \ .$$

(See Lemma A.2), Then, we find that for any positive integer m, $|[\frac{2}{2n+1}]\sum_{k\in\mathbf{I}_n}\sin\pi(k-\frac{1}{2})\times[\frac{[\cos\frac{\pi}{2n+1}(k-\frac{1}{2})]^2}{\sin\frac{\pi}{2n+1}(k-\frac{1}{2})}]| < 1$ and $\mathbf{1}_n^{'}\mathbf{P}_n\mathbf{Q}_n^{(m_1,m_2)}\mathbf{P}_n\mathbf{e}_1$ is less than one.
Similarly, for the second term of (32), we use

$$[1-\mathbf{1}_n^{'}\mathbf{P}_n^{*'}\mathbf{Q}_n^{(m_1,m_2)}\mathbf{P}_n^{*}\mathbf{e}_n] = \mathbf{1}_n^{'}\mathbf{P}_n^{*'}(\mathbf{I}_n-\mathbf{Q}_n^{(m_1,m_2)})\mathbf{P}_n^{*}\mathbf{e}_n \ .$$

Then, the second term of (32) becomes

$$[\sqrt{\frac{2}{n+\frac{1}{2}}}]^2\sum_{k=1}^{n}\sum_{j\in\mathbf{K}_n}[\sin\frac{2\pi}{2n+1}(k-\frac{1}{2})j\sin\frac{2\pi}{2n+1}(n-\frac{1}{2})j \ ,$$

which is less than 1. In this evaluation, we utilized the relations (i)

$$\begin{aligned}\sum_{k=1}^{n}\sin\frac{2\pi}{2n+1}(k-\frac{1}{2})j &= \frac{1}{2i}\frac{e^{i\frac{2\pi}{2n+1}jn}+e^{-i\frac{2\pi}{2n+1}jn}-2}{e^{i\frac{2\pi}{2n+1}j\frac{1}{2}}-e^{-i\frac{2\pi}{2n+1}j\frac{1}{2}}} \\ &= \frac{1}{2}\frac{1-\cos\frac{2\pi}{2n+1}jn}{\sin\frac{2\pi}{2n+1}j\frac{1}{2}} \ ,\end{aligned} \tag{33}$$

(ii) $1-\cos\frac{2\pi}{2n+1}jn = 1-\cos\pi j\cos\frac{\pi}{2n+1}j = 1-(-1)^j\cos\frac{\pi}{2n+1}j$ and (iii) for $2(n-1/2) = (2n+1)-2$, $\sin\frac{2\pi}{2n+1}(n-\frac{1}{2})j = [-\cos\pi j]\sin\frac{2\pi}{2n+1}j = 2[-\cos\pi j][\sin\frac{\pi}{2n+1}j][\cos\frac{\pi}{2n+1}j]$.
Then, the second term becomes

$$[\frac{4}{2n+1}]\sum_{j=\mathbf{K}_n}(-1)^{j+1}[1-(-1)^j\cos\frac{\pi}{2n+1}j]\cos\frac{\pi}{2n+1}j \ .$$

Since for any odd number j and $1 \leq j \leq n$, $[1+\cos\frac{\pi}{2n+1}j]\cos\frac{\pi}{2n+1}j > [1-\cos\frac{\pi}{2n+1}(j+1)]\cos\frac{\pi}{2n+1}(j+1)$, for any even muber m, we have $\sum_{j=1}^{m}(-1)^{j+1}[1-(-1)^j\cos\frac{\pi}{2n+1}j]\cos\frac{\pi}{2n+1}j < 0$.
Then, by using the fact

$$[\sqrt{\frac{2}{n+\frac{1}{2}}}]^2\sum_{k=1}^{n}\sum_{j=1}^{n}[\sin\frac{2\pi}{2n+1}(k-\frac{1}{2})j\sin\frac{2\pi}{2n+1}(n-\frac{1}{2})j = 1$$

(see Lemma A.2), $\mathbf{1}_n'\mathbf{P}_n^*(\mathbf{I}_n - \mathbf{Q}_n^{(m_1,m_2)})\mathbf{P}_n^{*\prime}\mathbf{e}_1$ is less than one.
Because each term of (32) is less than one, we have $|a_{2n}| < 1$. Then, we have convergence of T_{2k+1} as $k \to \infty$.

(**Part II**) We can apply the similar arguments to T_{2k} ($k \geq 1$). By using the recursive relations, for $k \geq 1$ we can represent

$$T_{2k} = A_{1*}^{(m_1,m_2)} + A_{2*}^{(m_1,m_2)}T_{2(k-1)+1}\ , \tag{34}$$

where $A_{1*}^{(m_1,m_2)}$ and $A_{2*}^{(m_1,m_2)}$ are $n \times n$ matrices as defined in Theorem 3.1. By evaluating the eigenvalues of $A_{2*}^{(m_1,m_2)}$, we find that the absolute value of eigenvalues are less than one, and we have convergence of T_{2k} as $k \to \infty$.
(**Q.E.D.**)

Finally, we mention to Theorem 5.3 of Kunitomo and Sato (2025) for convenience, which have been referred to several times in Setion 5. The notations here are slightly different from the original ones, and $\mathbf{v}_i$ and $\mathbf{v}_i^{(x)}$ in Kunitomo and Sato (2025) should be read as $\mathbf{e}_i^{(v)}$ and $\mathbf{e}_i^{(x)}$, respectively. We implicitly assumed in (1)-(4) with the restrictions on the coefficients of MA representation in the paper.

Theorem 5.3*: Assume that the fourth order moments of each element of $\mathbf{e}_i^{(x)}$ and $\mathbf{e}_i^{(v)}$ are bounded. Let

$$\hat{\mathbf{\Sigma}}_x(=(\hat{\sigma}_{gh}^{(x)})) = \frac{1}{m}\sum_{k=1}^{m}\mathbf{z}_k\mathbf{z}_k'\ , \tag{35}$$

which is the same as (39) in SK (2026).
Then
(i) For $m_n = [n^\alpha]$ and $0 < \alpha < 1$, as $n \longrightarrow \infty$

$$\hat{\mathbf{\Sigma}}_x - \mathbf{\Sigma}_x \xrightarrow{p} \mathbf{O}\ . \tag{36}$$

(ii) We set $\mathbf{\Sigma}_x = (\sigma_{gh}^{(x)})$. For $m_n = [n^\alpha]$ and $0 < \alpha < 0.8$, as $n \longrightarrow \infty$

$$\sqrt{m_n}\left[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}\right] \xrightarrow{\mathcal{L}} N\left(0, \sigma_{gg}^{(x)}\sigma_{hh}^{(x)} + \left[\sigma_{gh}^{(x)}\right]^2\right)\ . \tag{37}$$

The covariance of the limiting distributions of $\sqrt{m_n}[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}]$ and $\sqrt{m_n}[\hat{\sigma}_{kl}^{(x)} - \sigma_{kl}^{(x)}]$ is given by $\sigma_{gk}^{(x)}\sigma_{hl}^{(x)} + \sigma_{gl}^{(x)}\sigma_{hk}^{(x)}$ $(g, h, k, l = 1, \cdots, p)$.

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